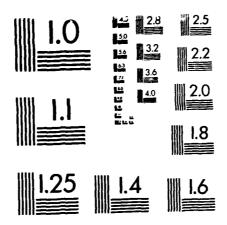
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BY

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A new concept of generalized normal equations is introduced for the simultaneous estimation of fixed parameters, random effects and random error. All the results are deduced from a general lemma on an optimization problem. This paper is self contained as all the algebraic results used are stated and proved. The unified theory developed in an earlier paper (Rao, 1988) is somewhat simplified.

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### A UNIFIED APPROACH TO ESTIMATION IN LINEAR MODELS WITH FIXED AND MIXED EFFECTS

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## A UNIFIED APPROACH TO ESTIMATION IN LINEAR MODELS WITH FIXED AND MIXED EFFECTS

Вy

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#### Abstract

A unified approach is developed for the estimation of unknown fixed parameters and prediction of random effects in a mixed Gauss-Markoff linear model. It is shown that both the estimators and their mean square errors can be expressed in terms of the elements of a g-inverse of a partitioned matrix which can be set up in terms of the matrices used in expressing the model. No assumptions are made on the ranks of the matrices involved. The method is parallel to the one developed by the author in the case of the fixed effects Gauss-Markoff model using a g-inverse of a partitioned matrix (Rao 1971, 1972, 1973, 1985).

A new concept of generalized normal equations is introduced for the simultaneous estimation of fixed parameters, random effects and random error. All the results are deduced from a general lemma on an optimization problem. This paper is self contained as all the algebraic results used are stated and proved. The unified theory developed in an earlier paper (Rao, 1988) is somewhat simplified.

Key words and phrases: g-inverse, Gauss-Markoff model with fixed and mixed effects, IPM (inverse partitioned matrix) method, normal equations.

AMS classification index: 62J05

#### 1. INTRODUCTION

The Gauss-Markoff model with fixed and random effects, called the mixed linear model, is written in the form

$$\mathbf{Y} = \mathbf{X}\mathbf{\beta} + \mathbf{U}\mathbf{\xi} + \mathbf{\varepsilon} \tag{1.1}$$

where Y is an n-vector of observations, X is a given  $n \times m$  matrix,  $\beta$  is an m-vector of unknown fixed parameters, U is a given nxp matrix,  $\xi$  is a p-vector of hypothetical random variables. We make the following assumptions on the first and second order moments of  $\xi$  and  $\xi$ .

$$E(\xi) = A \gamma$$
,  $E(\xi) = 0$ ,  $D(\xi) = \Gamma$ ,  $D(\xi) = G$ ,  $Cov(\xi, \xi) = 0$ . (1.2)

We refer to the model (1.1) - (1.2) as the GM(M) model where M within brackets refers to mixed effects. The corresponding model with fixed effects only, i.e., without the term  $U\xi$ , will be referred to as GM(F) when a distinction has to be made or simply as the GM model as it is usually known.

We develop a simple and a unified appproach in the general case, when nothing is assumed about the ranks of the matrices involved, for the estimation of the fixed parameter  $\beta$  and the prediction (or estimation) of the hypothetical variables  $\xi$  and  $\xi$ , when the other parameters  $\gamma$ ,  $\Gamma$  and G are partly known or completely known. First we prove a few algebraic lemmas. The following notations are used.

R<sup>n</sup>: n dimensional Euclidean real vector space.

R(Z): vector space spanned by the column vectors of the matrix Z.

 $\rho(\mathbf{Z})$ : rank of the matrix  $\mathbf{Z}$ .

 $Z^{\perp}$ : a matrix of maximum rank such that  $Z'Z^{\perp} = 0$ .

 $Z^-$ : a g-inverse of Z, i.e., a matrix satisfying the equation  $Z Z^- Z = Z$ .

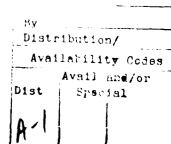
tr Z: sum of the diagonal elements of Z when it is a square matrix.

(A:B): the matrix obtained by adjoining the columns of the matrix B to

those of A.

We need the following results on g-inverses.





Lemma 1. Let Z be a g-inverse as defined above. Then:

(i) 
$$\mathbf{a} = \mathbf{Z}^{-} \mathbf{b}$$
 is a solution of the consistent equation  $\mathbf{Z}\mathbf{a} = \mathbf{b}$ . (1.3)

(ii) 
$$\rho(\mathbf{Z}^{\mathsf{T}}\mathbf{Z}) = \rho(\mathbf{Z}) = \operatorname{tr}(\mathbf{Z}^{\mathsf{T}}\mathbf{Z}).$$
 (1.4)

<u>Proof.</u> Since Z = b is consistent,  $b \in R(Z)$ , i.e., b = Zc for some c.

Then  $ZZ^{-}b = ZZ^{-}Zc = Zc = b$  which shows that  $Z^{-}b$  is a solution.

Now  $ZZ^{-}ZZ^{-} = ZZ^{-}$ , i.e.,  $ZZ^{-}$  is idempotent so that (1.4) follows.

Lemma 2. Let G be an n.n.d. (non-negative definite) matrix of order nxn, X be an nxm matrix and

$$\begin{pmatrix} \mathbf{G} & \mathbf{X} \\ \mathbf{X} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & -\mathbf{C}_4 \end{pmatrix} \tag{1.5}$$

be any choice of g-inverse. Then:

(i) 
$$X'C_1G = X'C_1'G = 0$$
,  $X C_2'X = X = X C_3 X$ . (1.6)

(ii) 
$$GC_1G = GC_1G =$$

(iii) 
$$X'C_1X = X'C_1X = 0$$
. (1.8)

(iv) tr 
$$\mathbf{G} \ \mathbf{C}_1 = \rho(\mathbf{G}:\mathbf{X}) - \rho(\mathbf{X}).$$
 (1.9)

<u>Proof</u> First we show that the equation

$$\mathbf{G} \mathbf{a} + \mathbf{X} \mathbf{b} = \mathbf{G} \lambda$$

$$\mathbf{X}' \mathbf{a} = \mathbf{X}' \mu \tag{1.10}$$

is consistent for any vectors  $\lambda$  and  $\mu$ . Let  $(\alpha': \beta')$  be a row vector such that

$$(\alpha':\beta')\begin{pmatrix} G & X \\ & & \\ X' & 0 \end{pmatrix} = 0 \Rightarrow \alpha'G + \beta'X' = 0, \quad \alpha'X = 0$$

$$\Rightarrow \alpha'G \alpha + \beta'X'\alpha = 0 = \alpha'G \alpha - 0 \Rightarrow \alpha'G = 0.$$

The last step follows since G is an n.n.d. matrix. Then,

$$(\alpha':\beta')\begin{pmatrix} \mathbf{G} & \lambda \\ & & \\ \mathbf{X'} & \mu \end{pmatrix} = \mathbf{o} \Rightarrow \begin{pmatrix} \mathbf{G} & \lambda \\ & & \\ \mathbf{X'} & \mu \end{pmatrix} \in \mathbf{R}\begin{pmatrix} \mathbf{G} & \mathbf{X'} \\ & & \\ \mathbf{X} & \mathbf{0} \end{pmatrix}$$

which establishes the consistency of (1.10). In such a case, using the giverse (1.5), we find a solution of (1.10).

$$\hat{\mathbf{a}} = \mathbf{C}_1 \mathbf{G} \lambda + \mathbf{C}_2 \mathbf{X}' \mu$$
,  $\hat{\mathbf{b}} = \mathbf{C}_3 \mathbf{G} \lambda - \mathbf{C}_4 \mathbf{X}' \mu$ .

Substituting  $\hat{a}$  for a in the second equation of (1.10) and equating the terms involving  $\lambda$  and  $\mu$  on both sides we obtain

$$X'C_1G = 0, X'C_2X' = X'.$$
 (1.11)

Further, the transpose of the g-inverse in (1.5) is also a g-inverse in view of the symmetry of the left hand side matrix of (1.5), and results analogous to (1.11) hold giving

$$X'C_{1}'G = 0, X'C_{3}'X' = X'.$$
 (1.12)

which prove (1.6). Again, substituting  $\hat{a}$  and  $\hat{b}$  for a and b in the first equation of (1.5) and equating the terms in  $\lambda$  on both sides, we have

$$G C_1 G + X C_3 G = G.$$
 (1.13)

Multiplying (1.13) by G  $C_1$  and G  $C_1'$  and using (1.11) and (1.12), we get the equalities in (1.7).

It is easy to see that

$$G \mathbf{a} + X\mathbf{b} = X\mu$$

$$X'\mathbf{a} = \mathbf{0}$$
(1.14)

is a consistent equation for any  $\mu$ , so that  $\hat{\mathbf{a}} = \mathbf{C}_1 \mathbf{X} \mu$  is a solution. Substituting  $\hat{\mathbf{a}}$  for  $\mathbf{a}$  in the second equation of (1.14), we find that  $\mathbf{X}' \mathbf{C}_1 \mathbf{X} \mu = \mathbf{o} \ \forall \ \mu \Rightarrow \mathbf{X}' \mathbf{C}_1 \mathbf{X} = \mathbf{o} = \mathbf{X}' \mathbf{C}_1 \mathbf{X}$ , which proves (1.8).

Now

$$\rho\begin{pmatrix} \mathbf{G} & \mathbf{X} \\ \mathbf{X'} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & -\mathbf{C}_4 \end{pmatrix} = \operatorname{tr}\begin{pmatrix} \mathbf{G}\mathbf{C}_1 + \mathbf{X}\mathbf{C}_3 & \mathbf{G}\mathbf{C}_2 - \mathbf{X}\mathbf{C}_4 \\ \mathbf{X'}\mathbf{C}_1 & \mathbf{X'}\mathbf{C}_2 \end{pmatrix}$$
$$= \operatorname{tr}(\mathbf{G}\mathbf{C}_1 + \mathbf{X}\mathbf{C}_3) + \operatorname{tr}\mathbf{X'}\mathbf{C}_2$$
$$= \operatorname{tr}\mathbf{G}\mathbf{C}_1 + \rho(\mathbf{X}\mathbf{C}_3) + \rho(\mathbf{X'}\mathbf{C}_2)$$
$$= \operatorname{tr}\mathbf{G}\mathbf{C}_1 + \rho(\mathbf{X}) + \rho(\mathbf{X'}) \qquad (1.15)$$

since XC<sub>3</sub> and X'C<sub>2</sub> are idempotent. But

$$\rho \begin{pmatrix} \mathbf{G} & \mathbf{X} \\ \mathbf{X'} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{C_1} & \mathbf{C_2} \\ \mathbf{C_3} & -\mathbf{C_4} \end{pmatrix} = \rho \begin{pmatrix} \mathbf{G} & \mathbf{X} \\ \mathbf{X'} & 0 \end{pmatrix} = \rho(\mathbf{G}:\mathbf{X}) + \rho(\mathbf{X}). (1.16)$$

Equating (1.15) and (1.16), we have (1.9) and Lemma 2 is proved. Now we prove the main lemma.

<u>Lemma 3</u> Let G and X be as in Lemma 2,  $g \in R(G : X)$  and  $p \in R(X')$ . Then

min (a G a + 2 a'g) = a' G a + 2 a' g  

$$X'a = p$$
 (1.17)

where  $a_*$  is any solution to

$$G a + X b = -g$$

$$X'a = p.$$
(1.18)

With C<sub>1</sub>, C<sub>2</sub>, C<sub>3</sub>, C<sub>4</sub> as defined in Lemma 2, one choice of the solution for

(a, b) of (1.18) is

$$\mathbf{a}_* = -\mathbf{C}_1 \, \mathbf{g} + \mathbf{C}_2 \, \mathbf{p}, \quad \mathbf{b}_* = -\mathbf{C}_3 \, \mathbf{g} - \mathbf{C}_4 \, \mathbf{p}$$
 (1.19)

giving the expressions for the minimum in (1.17) as

$$g'a_* - p'b_* = -g'C_1g + g'(C_2 + C_3)p + p'C_4p.$$
 (1.20)

<u>Proof</u> Let  $a_*$ ,  $b_*$  be any solution of (1.18), and  $Z = X^{\perp}$ . Then multiplying the first equation of (1.18), with (a, b) replaced by  $(a_*, b_*)$ , by Z' and  $a_*'$  we obtain

$$\mathbf{Z}' \mathbf{G} \mathbf{a}_* + \mathbf{Z}' \mathbf{g} = \mathbf{o} \tag{1.21}$$

$$\mathbf{a}_{*}\mathbf{G} \mathbf{a}_{*} + \mathbf{a}_{*}\mathbf{g} = -\mathbf{b}_{*}\mathbf{p} \tag{1.22}$$

A general solution of X'a = p is  $a_* + Z d$  where d is arbitrary. Then writing  $a = a_* + Z d$ ,

$$a'G \ a + 2 \ a'g = a + G \ a_* + 2 \ a + g + d'Z'G \ Z \ d + 2d'(Z'G \ a_* + Z'g)$$

$$= a + G \ a_* + 2 \ a + g + d'Z'G \ Z \ d, \quad using (1.21)$$

$$\geq a \cdot Ga_* + 2 \cdot a \cdot g = a \cdot g - b \cdot p$$
, using (1.22)

which proves (1.17). The result (1.20) is obtained by substituting the expressions (1.19) in (1.22).

Lemma 3 plays a crucial role in estimation and prediction problems in linear models. The results are given in Section 2.

<u>Lemma 4</u>. If Y is an n-vector random variable with E(Y) = 0 and  $D(Y) = \sigma^2 G$ , then an unbiased estimator of  $\sigma^2$  is

$$\sigma^2 = \frac{\mathbf{Y}'\mathbf{G}^{-}\mathbf{Y}}{\rho(\mathbf{G})} \tag{1.23}$$

where  $G^-$  is any inverse of G.

<u>Proof</u>

$$E(Y'G^{-}Y) = E \operatorname{tr}(G^{-}YY') = \sigma^{2} \operatorname{tr} G^{-}G = \sigma^{2}\rho(G)$$
, using (1.4).

#### 2. FIXED FFFECTS LINEAR MODEL

The Gauss-Markoff linear model with fixed effects is

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \mathbf{E}(\boldsymbol{\varepsilon}) = \mathbf{o}, \quad \mathbf{D}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{G}$$
 (2.1)

and the associated problems are those of estimating the unknown parameters  $\beta$  and  $\sigma^2$  and the random error  $\epsilon$ . We use the results of Lemma 3 in solving these problems. We denote

$$\begin{pmatrix}
\mathbf{G} & \mathbf{X} \\
\mathbf{X}' & 0
\end{pmatrix} = \begin{pmatrix}
\mathbf{C}_1 & \mathbf{C}_2 \\
\mathbf{C}_3 & -\mathbf{C}_4
\end{pmatrix} (2.2)$$

for any choice of the g-inverse.

#### 2.1 BLUE of p'β

Consider a linear function a Y as an unbiased estimator of  $p \beta$ . Then

$$E(\mathbf{a}'\mathbf{Y}) = \mathbf{a}'\mathbf{X}\boldsymbol{\beta} = \mathbf{p}'\boldsymbol{\beta} \ \forall \ \boldsymbol{\beta} \Rightarrow \mathbf{X}'\mathbf{a} = \mathbf{p}. \tag{2.3}$$

We find a by minimizing

$$V(\mathbf{a}'\mathbf{Y}) = \sigma^2 \mathbf{a}' \mathbf{G} \mathbf{a} \text{ subject to } \mathbf{x}'\mathbf{a} = \mathbf{p}. \tag{2.4}$$

Applying Lemma 3 with g = o, the BLUE of  $p'\beta$  is

$$\mathbf{a} \cdot \mathbf{Y} = \mathbf{p}' \mathbf{C} \cdot \mathbf{Y} \tag{2.5}$$

with the minimum variance

$$\sigma^2 \mathbf{a} \cdot \mathbf{G} \mathbf{a}_* = -\sigma^2 \mathbf{b} \cdot \mathbf{p} = \sigma^2 \mathbf{p} \cdot \mathbf{C}_4 \mathbf{p}$$
 (2.6)

using (1.20) and the expressions for  $\mathbf{a_*}$  and  $\mathbf{b_*}$  in (1.19).

#### 2.2 Estimation (Prediction) of &

Consider a linear function  $q' \mathcal{E}$  of  $\mathcal{E}$  and let a' Y, with  $E(a' Y) = o \Rightarrow a' X = o$ , be its predictor. Then the mean square error of prediction is

$$E(\mathbf{q}'\mathcal{E} - \mathbf{a}'\mathbf{Y})^2 = E(\mathbf{q}'\mathcal{E} - \mathbf{a}'\mathcal{E})^2$$

$$= \sigma^2(\mathbf{a}'\mathbf{G} \mathbf{a} - 2\mathbf{a}'\mathbf{G}\mathbf{q} + \mathbf{q}'\mathbf{G}\mathbf{q}). \tag{2.7}$$

Applying (1.19) with  $\mathbf{g} = -\mathbf{G}\mathbf{q}$  and  $\mathbf{p} = \mathbf{o}$ , the minimum of (2.7) is attained when the predictor is

$$a_{*}Y = -(C_{1}g)'Y = q'GC_{1}Y = q' \& .$$
 (2.8)

The minimum mean square error of prediction is, using (1.19) and (1.20),

$$\sigma^2(\mathbf{a} \cdot \mathbf{g} - \mathbf{b} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{G} \mathbf{q})$$

$$= \sigma^2(-\mathbf{q}'\mathbf{G}\mathbf{C}_1'\mathbf{G}\mathbf{q} + \mathbf{q}'\mathbf{G}\mathbf{q}) = \sigma^2\mathbf{q}'(\mathbf{G} - \mathbf{G}\mathbf{C}_1'\mathbf{G})\mathbf{q}. \tag{2.9}$$

The results (2.8) and (2.9) which hold for any q imply that the minimum dispersion error predictor of  $\mathcal{E}$  is

$$\hat{\mathbf{E}} = \mathbf{GC}_{1} \mathbf{Y} = \mathbf{GC}_{1} \mathbf{Y} \tag{2.10}$$

with

$$D(\hat{\mathcal{E}} - \mathcal{E}) = \sigma^2(\mathbf{G} - \mathbf{G}\mathbf{C}_1\mathbf{G}) \tag{2.11}$$

#### 2.3 Estimation of $\sigma^2$

Now

E ( 
$$\hat{\mathbf{E}}$$
) = E(GC<sub>1</sub>Y) = GC<sub>1</sub>X $\beta$  =  $\mathbf{o}$ , by (1.6)  
D (  $\hat{\mathbf{E}}$ ) = D(GC<sub>1</sub>Y) =  $\sigma^2$ GC<sub>1</sub>GC  $\frac{1}{1}$ G  
=  $\sigma^2$ GC<sub>1</sub>G, by (1.7).

Then using Lemma 4, an unbiased estimator of  $\sigma^2$  is

$$\hat{\mathbf{E}}'(\mathbf{G}\mathbf{C}_1\mathbf{G})^{-}\hat{\mathbf{E}} = \mathbf{Y}'\mathbf{C}_1\mathbf{G}(\mathbf{G}\mathbf{C}_1\mathbf{G})^{-}\mathbf{G}\mathbf{C}_1\mathbf{Y}$$
 (2.12)

with a suitable divisor. Note that  $Y \in R(G:X)$ , so that  $Y = G\lambda + X$  for a suitable  $\lambda$  and  $\mu$ . Then

$$\mathbf{Y}'\mathbf{C}_{1}\mathbf{G}(\mathbf{G}\mathbf{C}_{1}\mathbf{G})^{\mathsf{T}}\mathbf{G}\mathbf{C}_{1}\mathbf{Y} = \lambda'(\mathbf{G}\mathbf{C}_{1}\mathbf{G})(\mathbf{G}\mathbf{C}_{1}\mathbf{G})^{\mathsf{T}}(\mathbf{G}\mathbf{C}_{1}\mathbf{G})\lambda. \tag{2.13}$$

Since the terms in X vanish, using (1.6). Hence, by the definition of g-inverse, (2.13) reduces to

$$\lambda' \mathbf{G} \mathbf{C}_1 \mathbf{G} \lambda = (\mu' \mathbf{X}' + \lambda' \mathbf{G}) \mathbf{C}_1 (\mathbf{X} \mu + \mathbf{G} \lambda), \text{ using (1.6)}$$
$$= \mathbf{Y}' \mathbf{C}_1 \mathbf{Y}. \tag{2.14}$$

Now

$$\begin{split} E(\mathbf{Y}'\mathbf{C}_1\mathbf{Y}) &= \text{tr } E(\mathbf{Y}\mathbf{Y}'\mathbf{C}_1) = \sigma^2 \text{ tr}(\mathbf{G} + \mathbf{X}\beta\beta'\mathbf{X}')\mathbf{C}_1 \\ &= \sigma^2 \text{ tr } \mathbf{G}\mathbf{C}_1 + \sigma^2 \text{ tr } \beta\beta'\mathbf{X}'\mathbf{C}'\mathbf{X} = \text{tr } \mathbf{G}\mathbf{C}_1, \text{ using } (1.8) \\ &= \sigma^2 \text{ tr } \mathbf{G}\mathbf{C}_1 = \sigma^2[\rho(\mathbf{G}:\mathbf{X}) - \rho(\mathbf{X})], \text{ using } (1.9). \end{split}$$

Then an unbiased estimator of  $\sigma^2$  is

$$\hat{\sigma} = \frac{\mathbf{Y}'\mathbf{C}_1\mathbf{Y}}{\rho(\mathbf{G}:\mathbf{X})-\rho(\mathbf{X})}$$
 (2.15)

#### 2.4 Normal equations

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The expressions for the estimates of  $p'\beta$ ,  $\sigma^2$  and  $\epsilon$  obtained in sections 2.1 - 23 suggest a more direct way of obtaining them by first solving the consistent set of equations

$$G\alpha + X\beta = Y$$
  
 $X'\alpha = 0.$  (2.16)

If  $(\hat{\alpha}, \hat{\beta})$  is a solution of (2.6), then we have the following:

- (i) The BLUE of an estimable function  $\mathbf{p}'\beta$  is  $\mathbf{p}'$   $\beta$ .
- (ii) The minimum dispersion error predictor of  $\mathcal{E}$  is  $\hat{\mathcal{E}} = \mathbf{G}$   $\hat{\alpha}$ .
- (iii) An unbiased estimator of  $\sigma^2$  is  $\hat{\sigma}^2 = \mathbf{Y}' \hat{\alpha}/[\rho(\mathbf{G}:\mathbf{X}) \rho(\mathbf{X})]$ .

We may call (2.16) as the generalized normal equations for the simultaneous estimation of  $\epsilon$  and  $\beta$ .

If  $\hat{\alpha}$  and  $\hat{\beta}$  are obtained through a g-inverse as defined in (2.2), then we automatically have the expressions for the precisions of the estimates:

$$V(\mathbf{p}' \ \hat{\boldsymbol{\beta}}) = \sigma^2 \mathbf{p}' \mathbf{C}_4 \mathbf{p}$$

$$D(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}) = \sigma^2 (\mathbf{G} - \mathbf{G} \mathbf{C}_1 \mathbf{G}).$$

When  $G^{-1}$  exists, we can write the equations (2.16) in terms of the unknowns  $\epsilon$  and  $\beta$  to be estimated in the form

$$\mathbf{\mathcal{E}} + \mathbf{X}\mathbf{\beta} = \mathbf{Y}$$

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{\mathcal{E}} = \mathbf{o} \tag{2.17}$$

using the relationship  $\mathcal{E} = \mathbf{G} \alpha$ . Thus, the equations (2.17) are the appropriate normal equations for  $\mathcal{E}$  and  $\beta$  when  $\mathbf{G}^{-1}$  exists. In such a case,

eliminating  $\mathcal{E}$  in the second equation using the first equation in (2.17), we have

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\,\mathbf{\beta} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{Y} \tag{2.18}$$

which is the usual normal equation for  $\beta$  only. If  $\beta$  is a solution of (2.18) then from (2.17),  $\delta = Y - X \beta$  the usual residual.

The equation (2.16), however, is a more natural one which is simple to set up without any initial computations and which does not involve any assumptions on the ranks of the matrices involved.

#### 2.5 Projection operator

The second normal equation of (2.16) implies that  $\alpha = \mathbf{Z} \delta$  where  $\mathbf{Z} = \mathbf{X}^{\perp}$  and  $\delta$  is arbitrary. Substituting for  $\alpha$  in the first normal equation of (2.16),

$$X\beta + GZ \delta = Y \tag{2.19}$$

which provides the decomposition of the observed Y as 'signal + noise' giving estimates of  $X\beta$  and  $\delta$ . Note that R(GZ) and R(X) are disjoint and  $Y \in R(G:X) = R(GZ:X)$  w.p.1. Hence the decomposition (2.19) is unique. If  $\beta$  and  $\delta$  is a solution of (2.19), then an estimate of  $p'\beta$  is  $p'\beta$  and of  $\delta$  is  $\delta = GZ\delta$ .

Since R(X) and R(GZ) are disjoint, although R(GZ:X) may not span the whole of  $R^n$ , there exist projection operators  $P_X$  and  $P_{GZ}$  onto R(X) and R(GZ) in terms of which Y can be decomposed as in (2.19). Then

$$X \beta = P_x Y$$
 and  $\epsilon = GZ \delta = P_{GZ} Y = (I-P_x) Y$ .

Rao (1979) gives a detailed discussion of generalized projection operators.

#### 3. MIXED EFFECTS LINEAR MODEL

The mixed effects linear model is of the form

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U}\boldsymbol{\xi} + \boldsymbol{\varepsilon} \tag{3.1}$$

with

$$E(\mathcal{E}) = 0, \quad E(\xi) = A \gamma$$

$$D(\mathcal{E}) = G$$
,  $Cov(\mathcal{E}, \xi) = o$ ,  $D(\xi) = \Gamma$ .

We write the model in an alternative form

$$\mathbf{Y} = \mathbf{X}_{\star} \mathbf{\beta}_{\star} + \mathbf{U} \mathbf{\eta} + \mathbf{\varepsilon} \tag{3.2}$$

where

$$X_* = (X:UA), \quad \beta_* = (\beta':\gamma'), \quad \eta = \xi - A\gamma.$$

#### 3.1 Estimation of a mixed effect

Let  $p'\beta_* + q'\eta$  be a mixed effect to be estimated. If c + a'Y is an unbiased estimator, then

$$E(c+a'Y-p'\beta_*-q'\eta) = o \Rightarrow c = o \text{ and } X'_*a = p.$$

The mean square error is

$$E[\mathbf{a}'(\mathbf{U}\,\mathbf{\eta} + \mathbf{\epsilon}) - \mathbf{q}'\,\mathbf{\eta}]^2$$

$$= \mathbf{a}'\mathbf{G}_*\mathbf{a} - 2\mathbf{a}'\mathbf{U}\,\mathbf{\Gamma}\mathbf{q} + \mathbf{q}'\,\mathbf{\Gamma}\mathbf{q}$$

where  $G_* = U \Gamma U' + G$ . Applying Lemma 3, the optimum a is a solution of the equation

$$G_* \mathbf{a} + X_* \mathbf{b} = \mathbf{U} \Gamma \mathbf{q}$$

$$X_*' \mathbf{a} = \mathbf{p} . \tag{3.3}$$

If

$$\begin{pmatrix} \mathbf{G}_{\star} & \mathbf{X}_{\star} \\ \mathbf{X}_{\star}' & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{1} & \mathbf{C}_{2} \\ \mathbf{C}_{3} & -\mathbf{C}_{4} \end{pmatrix}$$
 (3.4)

for any choice of the g-inverse, then the best linear estimator is a \*Y where

$$\mathbf{a}_* = \mathbf{C}_1 \mathbf{U} \Gamma \mathbf{q} + \mathbf{C}_2 \mathbf{p} \tag{3.5}$$

and the mean square error is, using (1.19) and (1.20),

$$-a \cdot U \Gamma q - b \cdot p + q' \Gamma q$$

$$= q'(\Gamma - \Gamma U'C_{1}U\Gamma)q + p'C_{4}p - p'(C_{2} + C_{3})U\Gamma q.$$
 (3.6)

Writing  $\mathbf{p}' = (\mathbf{p}_{1}', \mathbf{p}_{2}'),$ 

$$p'\beta_* + q'\eta = p_1'\beta + p_2'\gamma + q'\eta = p_1'\beta + (p_2'-q'A)\gamma + q'\xi$$
 (3.7)

we find that the formulas (3.5) and (3.6) cover all special cases of linear functions involving one or more of the parameters  $\beta$  and  $\gamma$  and the random variable  $\xi$ .

#### 3.2 Estimation of the random error

Let r'E be a linear function of the random error estimated by a'Y. The condition of unbiasedness implies that  $a'X_* = o$ . The mean square error is

$$E(\mathbf{a}'\mathbf{Y}-\mathbf{r}'\mathbf{E})^2 = \mathbf{a}'\mathbf{G}_*\mathbf{a} - 2\mathbf{a}'\mathbf{G}\mathbf{r} + \mathbf{r}'\mathbf{G}\mathbf{r}. \tag{3.8}$$

Applying Lemma 3, the minimum of (3.8) is attained when the estimator of  $\mathbf{r}'$  is  $\mathbf{a}_*\mathbf{Y}$  where  $\mathbf{a}_* = \mathbf{C}_1\mathbf{G}\mathbf{r}$ . The minimum mean square error is using (1.19),

$$\mathbf{r}'(\mathbf{G}\mathbf{-GC}_{1}'\mathbf{G})\mathbf{r}.\tag{3.9}$$

From the above expressions it follows that the minimum dispersion error estimate of  $\epsilon$  is

$$\hat{\mathbf{E}} = \mathbf{G}\mathbf{C}_1\mathbf{Y} \text{ with } \mathbf{D}(\hat{\mathbf{E}} - \mathbf{E}) = \mathbf{G} - \mathbf{G}\mathbf{C}_1\mathbf{G}. \tag{3.10}$$

#### 3.3 Normal equations

The expressions for the estimators obtained in Sections 3.1 and 3.2 suggest the following estimation procedure. We set up the generalized normal equations

$$\begin{pmatrix}
G_* & X & U A \\
X' & 0 & 0 \\
A'U' & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix} = \begin{pmatrix}
Y \\
0 \\
0
\end{pmatrix}$$
(3.11)

and obtain a solution  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\gamma}$ . Then the estimate of  $\eta$  is  $\hat{\eta} = \Gamma U'$   $\hat{\alpha}$  and of  $\mathcal{E}$  is  $\hat{\mathcal{E}} = G$   $\hat{\alpha}$ . The estimate of  $\mathbf{p}_1 \hat{\beta} + \mathbf{p}_2 \hat{\gamma}$  when estimable is  $\mathbf{p}_1 \hat{\beta} + \mathbf{p}_2 \hat{\gamma}$ .

Denoting  $(X:UA) = X_*$  and  $\beta'_* = (\beta', \gamma')$ , the equations (3.10) can be written as

$$(G + U\Gamma U') \alpha + X_*\beta_* = Y$$

$$X'_* \alpha = 0.$$
(3.12)

If  $G^{-1}$  and  $\Gamma^{-1}$  exist, then multiplying the first equation by  $X * G^{-1}$  and  $\Gamma^{-1}\Gamma U'G^{-1}$  and using the second equation, we obtain the two equations

$$X_{*}^{'}G^{-1}U\eta + X_{*}^{'}G^{-1}X\beta_{*} = X_{*}^{'}G^{-1}Y$$

$$(\Gamma^{-1} + U'G^{-1}U)\eta + U'G^{-1}X\beta_{*} = U'G^{-1}Y.$$
(3.13)

Henderson (1984) derived equations of the type (3.13) when A = 0. (See also Harville, 1976). The equations (3.12) provide estimators of  $\eta$  and  $\beta$  directly.

When G and  $\Gamma$  are not both non-singular or when G and/or  $\Gamma$  is a complicated matrix, other methods of solving the equations (3.11) could be explored.

The estimators of  $\xi$ ,  $\epsilon$ ,  $\beta$  and  $\gamma$  obtained in Sections 3.1-3.3 involve the matrices G and  $\Gamma$  which may not be known. In the simplest possible case G and  $\Gamma$  may be of the form  $\sigma_1^2 V_1$  and  $\sigma_2^2 V_2$  respectively, where  $V_1$  and  $V_2$  are known and  $\sigma_1^2$  and  $\sigma_1^2$  are unknown variance components. In such a case,  $\sigma_1^2$  and  $\sigma_2^2$  may have to be estimated using techniques such as the MINQE or maximum likelihood as described in Rao and Kleffe (1988). The estimates of  $\sigma_1^2$  and  $\sigma_2^2$  may be substituted for the unknown values in the expressions for the estimators of  $\xi$ ,  $\epsilon$ ,  $\beta$  and  $\gamma$ .

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